SYDNEY GRAMMAR SCHOOL

2014 Assessment Examination

# FORM VI
# MATHEMATICS EXTENSION 2

**Thursday 15th May 2014**

**General Instructions**
- Writing time — 1 hour 30 minutes
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

**Total — 55 Marks**
- All questions may be attempted.

**Section I – 7 Marks**
- Questions 1–7 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

**Section II – 48 Marks**
- Questions 8–11 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

**Collection**
- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eight.
- Write your candidate number on this question paper and submit it with your answers.

**Checklist**
- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 79 boys

**Examiner**
RCF

**SECTION I - Multiple Choice**

**Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.**

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**QUESTION ONE**

The remainder when $P(z) = z^3 - 3z^2 + 5z - i$ is divided by $(z - 2i)$ is: **1**

(A) $12 - 3i$ (B) $12 + i$ (C) $6 - 3i$ (D) $6 + i$

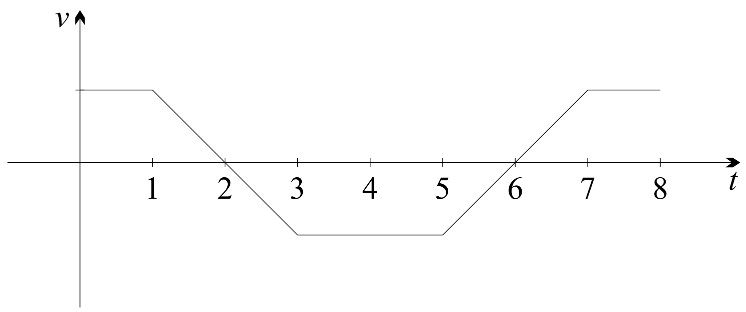
**QUESTION TWO**

A hyperbola is defined parametrically by the equations $x = 3\sec\theta$ and $y = \tan\theta$. **1**
Its eccentricity is:

(A) $\dfrac{2\sqrt{2}}{3}$ (B) $3$ (C) $\dfrac{\sqrt{10}}{3}$ (D) $\sqrt{3}$

**QUESTION THREE**

**1**




An object moves horizontally with velocity $v$ at time $t$ seconds as shown on the graph above. At which time does the object first return to its initial position?

(A) 2 s (B) 4 s (C) 6 s (D) 8 s

**QUESTION FOUR**

A particle is moving in simple harmonic motion along a straight line such that **1**

$$v^2 = 4(2 + 10x - x^2)$$

where $v$ is the velocity and $x$ is the displacement. The period $T$ and centre $x_0$ are:

(A) $T = \pi$ and $x_0 = 5$ (B) $T = 2\pi$ and $x_0 = -5$

(C) $T = \pi$ and $x_0 = -5$ (D) $T = 2\pi$ and $x_0 = 5$

**Exam continues next page ...**

**QUESTION FIVE**

Given the hyperbola $\dfrac{3x^2}{2} - \dfrac{2y^2}{3} = 1$, the gradient of the normal at the point $(1, \frac{\sqrt{3}}{2})$ is: **1**

(A) $\dfrac{3\sqrt{3}}{2}$ (B) $-2\sqrt{3}$

(C) $-\dfrac{2\sqrt{3}}{9}$ (D) $-\dfrac{3\sqrt{3}}{2}$

**QUESTION SIX**

A cannonball is fired on level ground with a given initial velocity. During its subsequent motion it behaves as a projectile accelerating under gravity but experiencing negligible air resistance. **1**

Which of the following statements about its subsequent motion is INCORRECT?

(A) At its maximum height the cannonball has velocity equal to the horizontal component of its initial velocity.

(B) The same range would be achieved using the complementary angle of projection.

(C) When it lands, the cannonball hits the ground with the same speed as it initially left the cannon.

(D) At its maximum height the cannonball experiences zero acceleration.

**QUESTION SEVEN**

It is known that the three roots of the cubic equation $x^3 + 2x^2 + 4x + 8 = 0$ form a geometric progression. The second term in this geometric progression is: **1**

(A) $2$ (B) $-2i$ (C) $-2$ (D) $2i$

——— End of Section I ———

**SECTION II - Written Response**

**Answers for this section should be recorded in the booklets provided.**

**Show all necessary working.**

**Start a new booklet for each question.**

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**QUESTION EIGHT** (12 marks) Use a separate writing booklet. **Marks**

(a) The equation $z^3 - 7z^2 + 17z - 15 = 0$ has a root $2 + i$.

(i) Explain why $2 - i$ is also a root of the equation. **1**

(ii) Find the third root of the equation. **1**

(b) The ellipse $\mathcal{E}$ has equation $3x^2 + 4y^2 = 12$.

(i) Show that the eccentricity of $\mathcal{E}$ is $\frac{1}{2}$. **1**

(ii) Find the coordinates of the foci $S$ and $S'$. **1**

(iii) Find the equations of the two directrices. **1**

(iv) Sketch $\mathcal{E}$, clearly showing its $x$ and $y$ intercepts, foci and directrices. **2**

(v) Find the equation of the tangent to $\mathcal{E}$ at the point $(1, \frac{3}{2})$. **2**

(c) Find the values of $k$ for which the polynomial $P(x) = x^3 + 4x^2 - 3x + k$ has a double zero. **3**

**Exam continues next page** ...

**QUESTION NINE** (12 marks) Use a separate writing booklet. **Marks**

(a) Suppose that $P(x) = x^3 + ax^2 + bx + 8$, where $a$ and $b$ are real.

(i) Show $P(4i) = (8 - 16a) + (4b - 64)i$ **1**

(ii) When $P(x)$ is divided by $x^2 + 16$, the remainder is $40 - 11x$. **2**
Find the values of $a$ and $b$.

(b) (i) The equation $x^3 - 4x + 6 = 0$ has roots $\alpha$, $\beta$ and $\gamma$. Find a polynomial equation with integer co-efficients that has roots $\alpha^2$, $\beta^2$ and $\gamma^2$. **2**

(ii) Hence, or otherwise, find the value of $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$. **2**

(c) A truck of mass 2000 kilograms driving along a straight road is propelled by a constant driving force from the engine of 1250 Newtons and experiences a resistive force of magnitude $2v^2$ where $v$ metres per second is the truck's velocity. The truck started from rest.

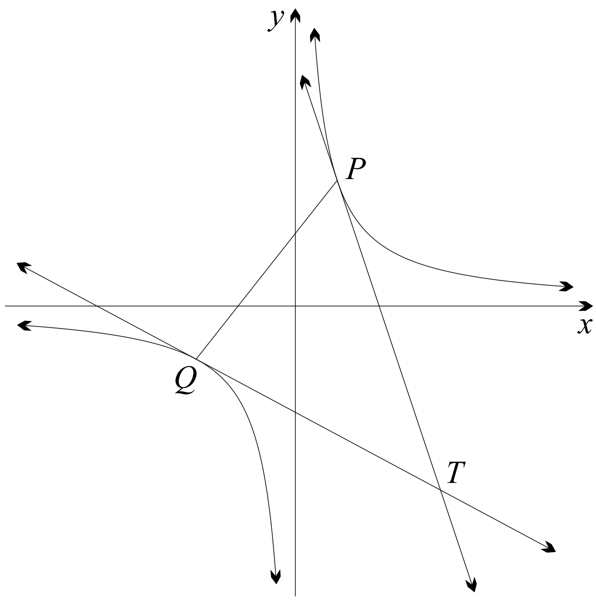
(i) Show that $\ddot{x} = \dfrac{625 - v^2}{1000}$. **1**

(ii) The truck approaches a maximum speed $U$. Find the value of $U$, giving your answer in metres per second. **1**

(iii) Find the distance it has travelled when it reaches a speed of $\dfrac{U}{2}$. Give your answer correct to the nearest metre. **3**

**Exam continues overleaf ...**

**QUESTION TEN** (12 marks) Use a separate writing booklet. **Marks**




(a) The points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ are on opposite branches of the rectangular hyperbola $xy = c^2$, where $c > 0$.

(i) Show that the equation of the chord $PQ$ is $x + pqy = c(p + q)$. **2**

(ii) The equation of the tangent at $P$ is $x + p^2y = 2cp$. Show that the tangents at $P$ and $Q$ intersect at the point $T\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$. **2**

(iii) The chord $PQ$ passes through the fixed point $(0, c)$. Describe the locus of $T$, including any restrictions. **3**

(b) Let $P(z) = 3z^8 - 10z^4 + 3$ and suppose that $\omega$ is a root of $P(z) = 0$.

(i) Show that $i\omega$ and $\dfrac{1}{\omega}$ are also roots of $P(z) = 0$. **2**

(ii) Show that $z = \sqrt[4]{3}$ is a root of $P(z) = 0$. **1**

(iii) Hence find all the roots of $P(z) = 0$. **2**

**QUESTION ELEVEN** (12 marks) Use a separate writing booklet. **Marks**

(a) A ball of mass 4 kilograms is thrown vertically upwards from the ground with an initial speed of 12 metres per second. The ball is subject to a downwards gravitational force of magnitude 40 Newtons and air resistance of $\frac{v^2}{5}$ Newtons in the opposite direction to the velocity $v$ metres per second.

(i) If we take the origin at ground level and upwards as the positive direction then the acceleration, until the ball reaches its highest point, is given by

$$\ddot{y} = -10 - \frac{v^2}{20}$$

where $y$ metres is its height. (Do NOT prove this.)

($\alpha$) Use the fact that $\ddot{y} = v\frac{dv}{dy}$, to show that, while the ball is rising, **2**

$$v^2 = 344e^{-\frac{y}{10}} - 200.$$

($\beta$) Hence find the maximum height reached in exact form. **1**

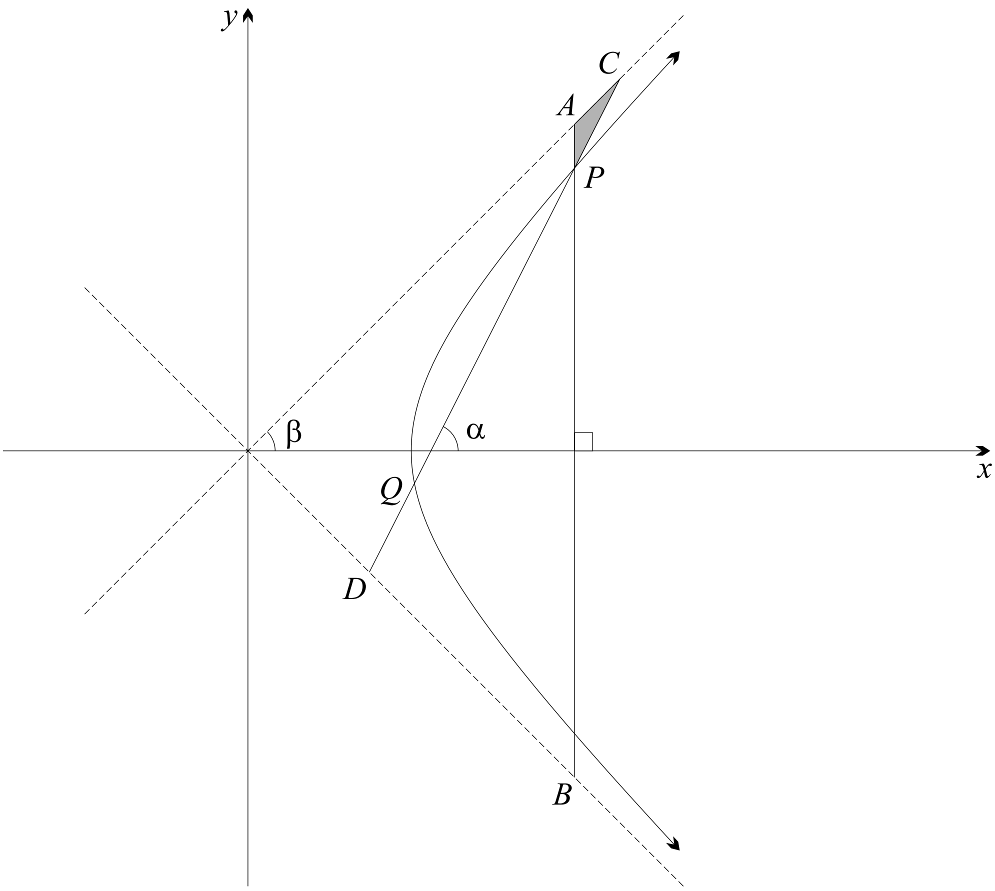
(ii) For the subsequent downwards motion, take downwards as the positive direction and define a new origin at the maximum height.

($\alpha$) Find the acceleration for the downwards motion. **1**

($\beta$) Hence find the speed at which the ball returns to the ground. Give your answer in metres per second, correct to two decimal places. **2**

**QUESTION ELEVEN** (Continued)

(b)




The point $P(a\sec\theta, b\tan\theta)$ is a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.
The line through $P$ perpendicular to the $x$-axis meets the asymptotes of the hyperbola $y = \dfrac{bx}{a}$ and $y = -\dfrac{bx}{a}$ at $A(a\sec\theta, b\sec\theta)$ and $B(a\sec\theta, -b\sec\theta)$ respectively.
A second line through $P$, with gradient $\tan\alpha$, meets the hyperbola again at $Q$ and meets the asymptotes at $C$ and $D$ as shown. The asymptote $y = \dfrac{bx}{a}$ makes a fixed angle $\beta$ with the positive $x$-axis at the origin.

(i) Show that $AP \times PB = b^2$. **1**

(ii) Use the sine rule in $\triangle ACP$ to show that $CP = \dfrac{AP\cos\beta}{\sin(\alpha - \beta)}$. **3**

Also show that $PD = \dfrac{PB\cos\beta}{\sin(\alpha + \beta)}$.

(iii) It follows from part (ii) that $CP \times PD$ is independent of $\theta$. Consequently its value does not dependent upon the choice of $P$. Hence deduce that $CP = QD$. **2**

End of Section II

**END OF EXAMINATION**

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1},\ \ n \neq -1;\ \ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ \ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax},\ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\text{NOTE}:\ \ln x = \log_e x,\ x > 0$$

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Candidate number: ...............

**Question One**

A ◯ B ◯ C ◯ D ◯

**Question Two**

A ◯ B ◯ C ◯ D ◯

**Question Three**

A ◯ B ◯ C ◯ D ◯

**Question Four**

A ◯ B ◯ C ◯ D ◯

**Question Five**

A ◯ B ◯ C ◯ D ◯

**Question Six**

A ◯ B ◯ C ◯ D ◯

**Question Seven**

A ◯ B ◯ C ◯ D ◯

# 4U MAY ASSESSMENT 2014

MULTI-CHOICE (1 MARK EACH)

① $P(z) = z^3 - 3z^2 + 5z - i$

$P(2i) = (2i)^3 - 3(2i)^2 + 5(2i) - i$

$= -8i + 12 + 10i - i$

$= 12 + i$ (B)

② $x = 3\sec\theta \quad y = \tan\theta$

$\frac{x^2}{3^2} - y^2 = 1$

$\therefore a = 3, b = 1$

$b^2 = a^2(e^2 - 1)$

$e^2 = 1 + \frac{b^2}{a^2}$

$e^2 = 1 + \frac{1}{9}$

$e = \frac{\sqrt{10}}{3}$ (C)

③ Area under velocity time graph represents displacement hence back to starting point when area above and below axis equal ie first @ 4s (B)

④ $v^2 = 4(2 + 10x - x^2)$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}(4 + 20x - 2x^2)$

$= 20 - 4x$

$= -4(x - 5)$

$\therefore n^2 = 4 \qquad x_0 = 5$

$n = 2$

$T = \frac{2\pi}{n} = \pi s$

(A)

⑤ $\frac{3x^2}{2} - \frac{2y^2}{3} = 1$

Diff wrt $x$ $\quad 3x - \frac{4y}{3}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{9x}{4y}$

Grad of normal is $\frac{-4y}{9x}$

$m = \frac{-4\left(\frac{\sqrt{3}}{2}\right)}{9} = \frac{-2\sqrt{3}}{9}$ (C)

⑥ Only incorrect statement is (D). Cannonball experiences constant acceleration due to gravity for entire motion.

⑦ $x^3 + 2x^2 + 4x + 8 = 0$

Roots in G.P.

Let roots be $\frac{a}{r}, a \,\&\, ar$

$\sum\alpha\beta\gamma = \frac{-8}{1}$

$\therefore a^3 = -8$

$\therefore a = (-2)$

Second term is $(-2)$

(C)

⑧ a) $z^3 - 7z^2 + 17z - 15 = 0$

(i) Since the equation has REAL CO-EFFICIENTS any complex roots appear in CONJUGATE PAIRS. ✓

(ii) $\sum\alpha = (2+i) + (2-i) + \alpha$

$= 4 + \alpha$

$= \frac{-(-7)}{1}$

$\therefore \alpha = 3$ ✓

OR $\sum\alpha\beta\gamma = \frac{-(-15)}{1}$

$\therefore (2+i)(2-i)\alpha = 15$

$5\alpha = 15$

$\alpha = 3$.

b) $3x^2 + 4y^2 = 12$

(i) $(\div 12)$

$\frac{x^2}{4} + \frac{y^2}{3} = 1$.

$a^2 = 4 \qquad a = 2$

$b^2 = 3 \qquad b = \sqrt{3}$.

$b^2 = a^2(1 - e^2)$

$\therefore e^2 = 1 - \frac{b^2}{a^2}$

$= 1 - \frac{3}{4}$

$e^2 = \frac{1}{4}$

$e = \frac{1}{2} \quad (e > 0)$ ✓ SHOW

(ii) Foci $S \,\&\, S'$ are $(ae, 0)$ and $(-ae, 0)$

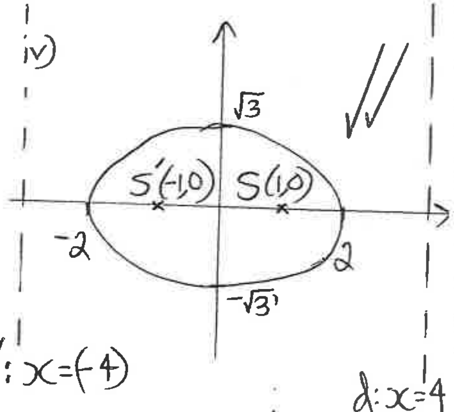
ie $(1, 0)$ and $(-1, 0)$ ✓

(iii) Directrices $x = \pm\frac{a}{e}$

$= \pm\frac{2}{\frac{1}{2}}$

$x = \pm 4$ ✓

iv)




✓✓

(v) $3x^2 + 4y^2 = 12$

Diff wrt $x$

$6x + 8y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-6x}{8y}$

$\therefore m = \frac{-6}{8 \times \frac{3}{2}}$

$= -\frac{1}{2}$ ✓

Eqn of tangent

$y - \frac{3}{2} = -\frac{1}{2}(x - 1)$

$2y - 3 = 1 - x$

$x + 2y - 4 = 0$. ✓

c) $P(x) = x^3 + 4x^2 - 3x + k$

$P'(x) = 3x^2 + 8x - 3$

$= (3x-1)(x+3)$ ✓

$P'(x) = 0 \Rightarrow x = \frac{1}{3}$ or $(-3)$ ✓

$P\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 + 4\left(\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right) + k$

$= \frac{1}{27} + \frac{4}{9} - 1 + k$

$= \frac{27k + 1 + 12 - 27}{27}$

$= \frac{27k - 14}{27}$ ✓

$P\left(\frac{1}{3}\right) = 0 \quad k = \frac{14}{27}$

$P(-3) = (-3)^3 + 4(-3)^2 - 3(-3) + k$

$= -27 + 36 + 9 + k$

$= 18 + k$

If $P(-3) = 0$, $k = (-18)$ ✓

⑨ $P(x) = x^3 + ax^2 + bx + 8$

a) (i) $P(4i) = (4i)^3 + a(4i)^2 + b(4i) + 8$

$= 64i^3 + 16i^2a + 4bi + 8$

$= -64i - 16a + 4bi + 8$

$= (8 - 16a) + (4b - 64)i$ ①

✓ SHOW

(ii) $P(x) = (x-4i)(x+4i)\,Q(x) + 40 - 11x$

$(x^2+16)\,Q(x) + 40 - 11x$

$\therefore P(4i) = 40 - 11 \times 4i$ ②

$\therefore$ equating real parts of ① + ②

$40 = 8 - 16a$

$16a = (-32)$

$a = (-2)$ ✓

equating imag parts

$4b - 64 = -44$

$4b = 20$

$b = 5.$ ✓

Alternative approach using long division algorithm

$$\begin{array}{r} x + a \\ x^2+16 \overline{)\, x^3 + ax^2 + bx + 8} \\ x^3 \quad\quad + 16x \quad - \\ \hline ax^2 + (b-16)x + 8 \\ ax^2 + 0x + 16a \quad - \\ \hline (b-16)x + 8 - 16a \end{array}$$

equating:

coeffs of $x$ $\therefore (b - 16) = -11 \Rightarrow b = 5$

constants $(8 - 16a) = 40 \Rightarrow a = (-2)$

b)(i) $P(x) = x^3 - 4x + 6 = 0$ has roots $\alpha, \beta, \gamma$

Eqn with roots $\alpha^2, \beta^2, \gamma^2$ requires replacing $x$ with $\sqrt{x}$

$P(\sqrt{x}) = (\sqrt{x})^3 - 4(\sqrt{x}) + 6$ ✓

$= \sqrt{x}(x-4) + 6$

$\therefore \sqrt{x}(x-4) + 6 = 0$

$\sqrt{x}(x-4) = -6$

(sq) (sq)

$x(x-4)^2 = 36$

$x(x^2 - 8x + 16) = 36$

$\underline{x^3 - 8x^2 + 16x - 36 = 0}$ ✓

has roots $\alpha^2, \beta^2, \gamma^2$

(ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2}{\alpha^2\beta^2\gamma^2}$ ✓

$= \frac{\sum \alpha\beta}{\sum \alpha\beta\gamma} = \frac{16/1}{-(-36/1)}$ ✓

$= \underline{4/9}$

Alternative Approaches forming NEW eqns with roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

a) Use $x^3 - 8x^2 + 16x - 36 = 0$ and replace $x$ with $\frac{1}{x}$

$\frac{1}{x^3} - \frac{8}{x^2} + \frac{16}{x} - 36 = 0$

$(\times x^3)$

$1 - 8x + 16x^2 - 36x^3 = 0$

$36x^3 - 16x^2 + 8x - 1 = 0$ then (ii) $\sum \alpha = -\frac{b}{a} = \frac{16}{36} = \frac{4}{9}$

β) Use original $x^3 - 4x + 6 = 0$ and replace $x$ with $\frac{1}{\sqrt{x}}$

$\left(\frac{1}{\sqrt{x}}\right)^3 - 4\left(\frac{1}{\sqrt{x}}\right) + 6 = 0$

$\times x^{3/2}$

$1 - 4x + 6x^{3/2} = 0$

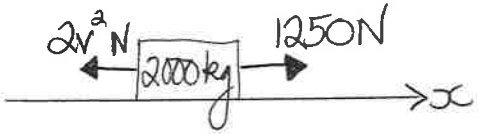
$6x^{3/2} = 4x - 1$

(sq) (sq)

$36x^3 = 16x^2 - 8x + 1$

$36x^3 - 16x^2 + 8x - 1 = 0$ as before

c)




(i) Eqn of motion →

$1250 - 2v^2 = 2000\ddot{x}$

$\ddot{x} = \frac{2(625 - v^2)}{2000}$ ✓ SHOW

$\ddot{x} = \frac{625 - v^2}{1000}$

(ii) Max Speed $\ddot{x} = 0$ $\therefore 625 - U^2 = 0$

$U^2 = 625$

$\underline{U = 25 \text{ m/s}}$ ✓ $(v > 0)$

(iii) $v\frac{dv}{dx} = \frac{625 - v^2}{1000}$

$\int_0^{U/2} \frac{2v}{625 - v^2} dv = \int_0^D \frac{1}{500} dx$ ✓ Initially $v = 0$ $x = 0$; when $v = \frac{U}{2}$ let $x = D$.

$\left[-\ln(625 - v^2)\right]_0^{U/2} = \left[\frac{x}{500}\right]_0^D$ ✓

$-\ln\left(625 - \frac{625}{4}\right) - \left(-\ln 625\right) = \frac{D}{500}$

$\ln\left(\frac{625}{625(3/4)}\right) = \frac{D}{500}$

$500 \ln\left(\frac{4}{3}\right) = D$

Distance travelled is approx. $\underline{144 \text{ metres}}$ ✓

Alternative approach using reciprocals and indefinite integration

$$\frac{dv}{dx} = \frac{625-v^2}{1000v}$$

$$\frac{dx}{dv} = \frac{1000v}{625-v^2}$$

$$\int dv \qquad \int dv$$

$$x = -500\ln(625-v^2)+C$$

$v=0$ $x=0$ $\qquad 0 = -500\ln 625 + C$

$$C = 500\ln 625$$

$$x = 500\ln\left(\frac{625}{625-v^2}\right)$$

$v = \frac{U}{2} = \frac{25}{2}$

$$= 500\ln\left(\frac{625}{625}\times\frac{1}{3/4}\right)$$

$$= 500\ln\frac{4}{3}$$

$$\doteqdot 144\text{ m}$$

(10)

a) (i) $m_{PQ} = \dfrac{\frac{c}{q}-\frac{c}{p}}{cq-cp}$

$$= \frac{c\left(\frac{p-q}{pq}\right)}{c(q-p)} \quad (p\neq q)$$

$$= -\frac{1}{pq} \quad ✓$$

$\therefore$ Eqn of chord $y-\frac{c}{p} = -\frac{1}{pq}(x-cp)$ ✓SHOW

$$pqy - cq = -x+cp$$

$$x+pqy = c(p+q)$$

(ii) Tangent at P $\quad x+p^2y = 2cp \quad (1)$

at Q $\quad x+q^2y = 2cq \quad (2)$

Solve simultaneously (1)+(2)

$$2cp - p^2y = 2cq - q^2y$$

$$2cp-2cq = (p^2-q^2)y$$

$$\frac{2c(p-q)}{(p-q)(p+q)} = y \quad ✓\text{SHOW}$$

$p\neq q$ since distinct points on opposite branches of R.

$$\therefore y = \frac{2c}{p+q}$$

sub into (1) $\quad x = 2cp - p^2\left(\frac{2c}{p+q}\right)$

$$x = \frac{2cp(p+q)-2cp^2}{(p+q)}$$

$$= \frac{2cpq}{p+q} \quad ✓\text{SHOW}$$

$$\therefore T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$$

(iii) If chord PQ passes through $(0,c)$ then $0+pqc=c(p+q)$

$c \neq 0 \therefore pq = p+q$ ✓

$p+q \neq 0$ since $p \neq 0, q \neq 0 \therefore pq \neq 0$.

$$T\left(\frac{2c(p+q)}{(p+q)}, \frac{2c}{p+q}\right)$$

$$= T\left(2c, \frac{2c}{pq}\right)$$

also $pq < 0$ since P and Q on opposite branches of hyperbola ✓

hence locus of T is vertical ray, with eqn $x = 2c$ ✓ where $y < 0$ ✓ Restriction

note: Chord PQ requires a positive gradient for Q to lie on opposite branch hence if P on first quadrant branch $0 < p < 1$

b) $P(z) = 3z^8 - 10z^4 + 3$

$P(w) = 0 \quad \therefore 3w^8 - 10w^4 + 3 = 0$

(i) $P(iw) = 3(iw)^8 - 10(iw)^4 + 3$

$= 3w^8 - 10w^4 + 3$ ✓SHOW (since $i^8 = i^4 = 1$)

$= 0$

$\therefore iw$ a root of $P(z) = 0$

$P\left(\frac{1}{w}\right) = 3\left(\frac{1}{w}\right)^8 - 10\left(\frac{1}{w^4}\right) + 3$

$= \frac{3}{w^8} - \frac{10}{w^4} + 3$

$w^8 \times P\left(\frac{1}{w}\right) = 3 - 10w^4 + 3w^8$ ✓SHOW

$= 0$

but $w \neq 0$ since $P(0) = 3$ hence 0 not a soln

$\therefore P\left(\frac{1}{w}\right) = 0 \quad \therefore \frac{1}{w}$ a root of $P(z) = 0$

(ii) $P(3^{\frac{1}{4}}) = 3(3^{\frac{1}{4}})^8 - 10(3^{\frac{1}{4}})^4 + 3$

$= 3 \times 3^2 - 10 \times 3 + 3$ ✓

$= 27 - 30 + 3$

$= 0$

$\therefore \sqrt[4]{3}$ is a root of $P(z) = 0$

since $P(z)$ has even symmetry $z = -\sqrt[4]{3}$ a root

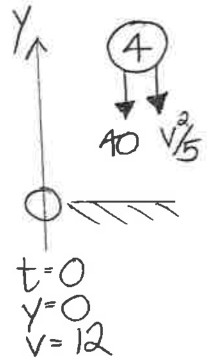
from (i) $z = \sqrt[4]{3}\,i$ also a root, and its conjugate $-\sqrt[4]{3}\,i$ since real coefficients

from (i) $z = \frac{1}{\sqrt[4]{3}}$ a root, also $z = \frac{-1}{\sqrt[4]{3}}$ ✓✓

also $z = \frac{i}{\sqrt[4]{3}}$ and $z = \frac{-i}{\sqrt[4]{3}}$

Hence eight roots are $\pm\sqrt[4]{3}, \pm\sqrt[4]{3}\,i, \pm\frac{1}{\sqrt[4]{3}}, \pm\frac{i}{\sqrt[4]{3}}$

(11) a)




(i) Eqn of motion

$\uparrow \; 4\ddot{y} = -40 - \frac{v^2}{5}$

$\ddot{y} = -10 - \frac{v^2}{20}$ (Given)

($\alpha$) $v\frac{dv}{dy} = -\left(\frac{200+v^2}{20}\right)$

$\int \frac{2v}{200+v^2}\,dv = \int -\frac{dy}{10}$ ✓

$\ln(200+v^2) = -\frac{y}{10} + C$

when $y=0$ $v=12$

$\therefore \ln 344 = 0 + C$ ✓ SHOW

$\therefore \ln(200+v^2) - \ln 344 = -\frac{y}{10}$

$\ln \frac{200+v^2}{344} = -\frac{y}{10}$

$\frac{200+v^2}{344} = e^{-y/10}$

$v^2 = 344e^{-y/10} - 200$ ∎

($\beta$) Max Height $v=0$

$\therefore 344e^{-y/10} = 200$

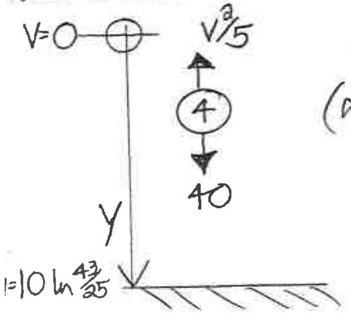
$e^{-y/10} = \frac{200}{344}$

$e^{y/10} = \frac{344}{200}$

$= \frac{43}{25}$

$y = 10\ln\frac{43}{25}$ metres ✓

(ii) DOWNWARD




Eqn of motion $\downarrow$

($\alpha$) $4\ddot{y} = 40 - \frac{v^2}{5}$

$\ddot{y} = 10 - \frac{v^2}{20}$ ✓

($\beta$) $v\frac{dv}{dy} = \frac{200-v^2}{20}$

$\int_0^V \frac{-2v}{200-v^2}\,dv = \int_0^{10\ln\frac{43}{25}} -\frac{dy}{10}$ initially $y=0$ $v=0$

at ground $y = 10\ln\frac{43}{25}$, speed $V$

$\left[\ln(200-v^2)\right]_0^V = \left[-\frac{y}{10}\right]_0^{10\ln\frac{43}{25}}$ ✓

$\ln(200-V^2) - \ln 200 = -\ln\frac{43}{25}$

$\ln(200-V^2) = \ln 200 - \ln\frac{43}{25}$

$= \ln\left(\frac{5000}{43}\right)$

$\therefore 200 - V^2 = \frac{5000}{43}$

$V^2 = \frac{8600}{43} - \frac{5000}{43}$

$= \frac{3600}{43}$

hence $V = \frac{60}{\sqrt{43}}$ m/s. ✓ $(V>0)$

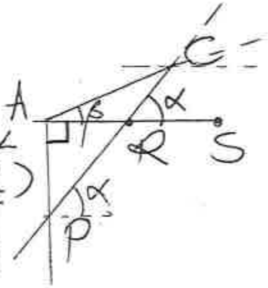
$\doteqdot 9.15$ m/s.

b) $AP = y_A - y_P$
$= b\sec\theta - b\tan\theta$
$= b(\sec\theta - \tan\theta)$

$PB = y_P - y_B$
$= b\tan\theta - (-b\sec\theta)$
$= b(\tan\theta + \sec\theta)$

$\therefore AP \times PB = b(\sec\theta - \tan\theta) \times b(\tan\theta + \sec\theta)$ ✓ SHOW

$= b^2(\sec^2\theta - \tan^2\theta)$
$= b^2$




(ii) Consider $\triangle ACP$
$\angle ACP = \alpha - \beta$ (Exterior $\angle$ $\triangle CAR$)
$\angle CAP = \beta + \frac{\pi}{2}$ (Adjacent Angles)

Using sine rule

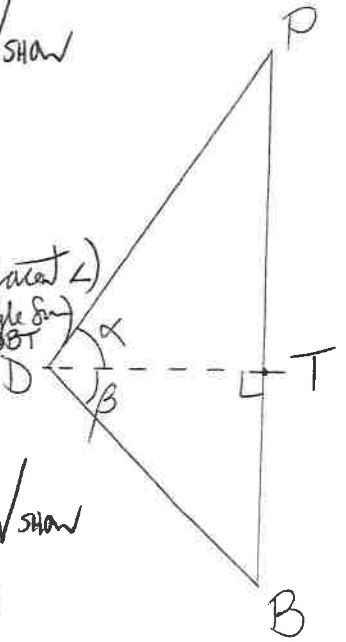
$$\frac{AP}{\sin(\angle ACP)} = \frac{CP}{\sin(\angle CAP)}$$ ✓

$$\therefore \frac{AP}{\sin(\alpha-\beta)} = \frac{CP}{\sin(\beta+\frac{\pi}{2})}$$

since $\sin(\beta+\frac{\pi}{2}) = \cos\beta$.
translating sine curve to left by $\frac{\pi}{2}$

$$\frac{AP}{\sin(\alpha-\beta)} = \frac{CP}{\cos\beta}$$ ✓ SHOW

$$\therefore CP = \frac{AP\cos\beta}{\sin(\alpha-\beta)}$$




Similarly in $\triangle PDB$
$\angle PDB = \alpha + \beta$. (Adjacent $\angle$)
$\angle PBD = \frac{\pi}{2} - \beta$. (Angle Sum $\triangle DBT$)

Using sine rule

$$\frac{PD}{\sin(\angle PBD)} = \frac{PB}{\sin(\angle PDB)}$$ ✓ SHOW

$$\frac{PD}{\sin(\frac{\pi}{2}-\beta)} = \frac{PB}{\sin(\alpha+\beta)}$$

$$PD = \frac{PB\cos\beta}{\sin(\alpha+\beta)}$$

(iii) $$CP \times PD = \frac{AP\cos\beta}{\sin(\alpha-\beta)} \times \frac{PB\cos\beta}{\sin(\alpha+\beta)}$$

$$= \frac{AP \times PB\cos^2\beta}{\sin(\alpha-\beta)\sin(\alpha+\beta)}$$

$$= \frac{b^2\cos^2\beta}{\sin(\alpha-\beta)(\alpha+\beta)}$$

$b, \beta$ constants
only variable $\alpha$

since product doesn't depend upon choice of P. ✓

$CP \times PD = CQ \times QD$
$CP \times (PQ + QD) = (CP + PQ) \times QD$ ✓ SHOW
$CP \times PQ + ~~CP \times QD~~ = ~~CP \times QD~~ + PQ \times QD$
$\therefore CP = QD$ ■ (since $PQ \neq 0$)